Bargaining in Networks

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Abstract

We analyse two models of stationary bargaining in networks with an arbitrarily small chance of death (leaving the game with zero payoff): In the first *unmatched leave* model for agents that are unmatched in a round and second in the *no agreement leave* model for agents that cannot agree on the proposed shares. Characterising equilibria for three examples, we show that uniqueness need not hold for the further. We compare equilibrium outcomes to the model of **?**. We show: A variant of the unmatched leave model with discounting as additional friction admits a unique equilibrium. In an appendix, we conclude equilibrium for a dynamic bargaining model as in **?** with an added threat of death of agents.

1 Introduction

Classic general equilibrium theory assumes large markets with no individual power. Given this assumption the study of allocations and prices dramatically simplifies—goods are perfectly divisible and homogeneous, similar products will have the same price and no trader will have market power. Many of the above stated conclusions fail to hold in reality. Traders do not take prices as given and potential trading partners are determined by relations, distance, and technology. The power of a trader in a network models of bargaining is determined by her trading partners - the agents she shares a link with—and also the global structure of the market and in the case of dynamic markets also on future and past of the market situation.

This brought interest in foundations of General Equilibrium Theory in models of bargaining. **?** contributed to the observation, that in models assuming so-called *cloning assumption* [**?**], competitive outcomes happen by pure chance: This is the assumption that successful bargaining partners are each replaced by clones. Equivalently, that the stock of agents in the game is exogenous. Such models are also called *stationary* models. Studying a model of bargaining where *inflow* of agents is exogenous, **?** shows outcomes are Walrasian. We call these models *dynamic* (which does not mean that no steady states might be sought).

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The distinction between stationary and dynamic models also divides the relatively young literature on bargaining on networks. We analyse models of stationary bargaining, but also show how these can, following **?**, be used to show equilibrium existence in a related dynamic model.

? serves as a starting point for our research on stationary models: He describes a bargaining game of stationary bargaining. One particularity of this model becomes evident in simple examples - although individual traders have market power, they can not fully discriminate their disadvantaged bargaining partners. In an example where a monopolist a may propose to and respond to proposals from players b and c, each four with equal probability, the model of **?** predicts equilibrium payoffs for patient agents (that is, vanishing discounting) of

$$(u_a, u_b, u_c) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}).$$

The setup will be discussed later for different models and is shown in Figure 1 on page 8. In this model, a can thus not use that she has monopoly power to extract all surplus.

We show that when replacing discounting with a small threat of death, we arrive at different conclusion, but that these models do not yield both equilibrium uniqueness and existence. More specifically, we discuss two different kinds of threat - one which threatens only unmatched players in each round (unmatched leave) and another one that threatens all that did not trade (no agreement leave). We consider, in examples, equilibrium existence and uniqueness when the death rate is vanishingly small.

Most papers on stationary bargaining in networks (e.g. [**?????**]) use fixed point equation systems for finding subgame-perfect equilibria, which is our main tool as well.

The plan of the paper is the following: Before defining variants of **?**'s model that for the example of players a, b, c from above do not contradict monopoly intuitions in section 3, we review related work in section 2. Following this, we characterise equilibria as is standard in the literature on stationary bargaining by a fixed point equation system in section 4. Using this, we characterise equilibria in easy examples in section 5. We show equilibrium existence and uniqueness for a related model in section 6. We conclude with section 7.

2 Literature Review

The study of network models of bargaining emerged from laying out foundations and challenging applicability for theories of General Equilibrium. The first step in this direction was done by game theorists fully describing the bargaining process between two traders. **?** is one of the pioneering papers in this field. Already **?** described a type of network in which each buyer was connected to each seller. They showed that the division of the surplus depends on the ratio of sellers to buyers. Other papers that build up on a dynamic description (dynamic in the sense of game theory) of bargaining situations with many traders are **?**,

??? and **?**. As already pointed out, the literature splits in two main parts of work: stationary models with endogenous inflow of agents - all that arrive at agreement in bargaining leave and are immediately replaced by clones - and with exogenous inflow of agents, where no replacement takes place.

? was among the first to highlight this difference: For the further, cloning shapes outcomes that are generically non-competitive. The endogenous inflow which fixes the stock of agents present in the game at any time to en exogenously determined composition in **?** and later in **?** is the key assumption which leads to the described steady states in which the short side of markets cannot fully exploit market power.

For the literature on stationary bargaining on networks, **?** is central. In his paper a link between $\{i, j\}$ of the undirected network is chosen with probability $p_{\{i,j\}}$ and with $p = 1/2$ either i or j is allowed to make a proposal on how to share a unit surplus. If the resp. other player agrees on this share, both leave the game and are be replaced by clones in round $t + 1$. Otherwise they remain. Discounting is exponential with rate $\delta \in (0, 1)$.

? shows that in the limit $\delta \to 1$ payoffs for all agents are uniquely determined by their position in the network and independent of probabilities $\mathbf{p}$. For his measurement of bargaining strength he introduces the so-called shortage ratio, generalising **?**. The shortage ratio is, for a set M of agents that are not able to bargain with one another (in the absence of links between them; think the set of sellers resp. buyers), and the set of all their possible trade partners N

$$\frac{|M|}{|N|}.$$

He shows that, iteratively considering sets of minimal shortage ratio, one can compute equilibrium payoffs for $\delta \to 1$ in finite time. He demonstrates with an example, that it might be the case, that in equilibrium some $\{i, j\}$ might never come to an agreement, and that this can even happen for links connecting market.

? presents an extension. In his model not only two agents, but also larger sets of agents can bargain with each other over a surplus. Each round a set within which is bargained and a proposer are randomly selected. Afterwards, the proposer makes sequential offers to all other partners in the coalition. Besides generalization, a main achievement of **?** is a characterization of equilibrium payoffs as solution of a quadratic programming problem and of bargaining power as dual variables in an optimal solution.

We also build upon dynamic models of bargaining. We use death rates/exogenous outflows in agents for a model of stationary bargaining as e.g. in **?**. We leave for further work a generalisation of the existence result of **?** for dynamic models, describe the proof idea however in section 7.

Other dynamic models of bargaining on networks mostly assumes the special case of (trivially exogenous) zero inflow of new agents: In the initial work of **?** either sellers or buyers post a price at period t, the respective opponent publicly posts a price at which she accepts. The matched partners that agree on a price leave the game and the remaining players proceed to the next round. Players are not replaced by clones here. **?** shows that this model has a subgame perfect equilibrium. [**?**] tested this model in the lab and confirmed most of the results. In contrast to the public offer in **?** the model by **?** uses an centralized matching mechanism to allocate the in the game remaining agents. So every round t, the efficient procedure matches a maximum of the remaining agents and afterwards decides with equal probability who is allowed to propose the split of the unit surplus.

3 Model

As noted in the literature review, in the literature, roughly two kinds of models are assumed:

- The matching technology is exogenous, the (in-/out-)flow of agents is endogenous. (stationary).
- The matching technology is endogenous, the (in-/out-)flow of agents is exogenous. (dynamic)

Only in the concluding section we consider the latter. The rest of the proposal, we focus on the stationary model.
We start with notations concerning our model of networks, a *graph*.

3.1 Graph

In our network V denotes the finite set of player types [1]. We define a (directed) graph $G = (V, E)$, with V a finite set of vertices and $E \subseteq \{(i,j)|i \neq j \in V\}$ the set of links. We call for a link $(i,j) \in E$, i its tail and j its head. We denote by $\delta^+(v) = \{\ (v,w) \in E \mid w \in V\}$ resp. $\delta^-(v) = \{(w,v) \in E \mid w \in V\}$ the outgoing resp. incoming links of v. Denote by $\delta(v) = \delta^+(v) \cup \delta^-(v)$ the set of links that either start or end in v. Having this notation, we can start with out first model. Despite being conceptually very similar to **?**, our model differs technically, as we assume bargaining on directed graph, which allows for more natural notation and added generality. To translate the model of **?** to ours, just replace a link $\{i,j\}$ in Menea by two links (i,j) and (j,i).

3.2 Exogenous Matching Technology with Death Rate

We assume that G is equipped with a probability mass function

$$p \colon E \to [0,1]$$

[1] for an interpretation that each $v \in V$ denotes a player type that a continuum of players of fixed size has, see either **?** or section 7

which we call the *matching probability*. We furthermore assume that there is a uniform *surplus* $s > 0^2$.

The bargaining game consists of infinitely many periods $t \in \mathbb{N}$. In the basic setup both s and p do not depend on t.[3] At each t, one link $e \in E$ is chosen with probability p_e. The tail of the link is called the *proposer*. She offers a division of the surplus s to the head of the link, the *responder*. The responder may accept or reject the offer. More specifically, an offer consists of $(x, s-x), x \in \mathbb{R}$, which allows for proposals of which the proposer can be sure they will be rejected. After agreement both players leave the game with the shares agreed on. If the responder rejects the offer, both players stay in the game for period $t+1$. Thus, after the bargaining stage of the game agents exit and enter the game.

In contrast to **?**, market frictions arise by introducing an additional possibility of exit after the bargaining stage. We assume a death rate

$$d \in [0, 1].$$

We consider two variants of the effect of d: The *unmatched leave* and the *no agreement leave* models.

In the further, agents that are in a round neither proposer nor responder leave the game with zero payoff with a probability of d. In the latter also agents that do not reach agreement leave the game with probability d.

We assume common knowledge of the network structure, i.e. G, p and s. Every agent has perfect information about the previous decisions in their vertices. We denote such a history by h and the set of histories by H. Additionally in period t, i proposes a share x of the surplus s to the responder j:

$$\sigma_i^{\mathrm{p}}\colon H \times E \to \mathbb{R}^+, \sigma_i(h, i \to j = x)$$

is the strategy by agent i given she is a proposer and

$$\sigma_i^{\mathrm{r}}\colon H \times E \times \mathbb{R}^+ \to \{\text{agree}, \text{not agree}\}$$

if she is a responder. We allow for mixed strategies.

Such a strategy is a *subgame perfect equilibrium* for the proposed game if it is a Nash equilibrium at every stage and given any history. In most parts of our analysis we are interested in a stationary concept. Thus, we call a strategy profile $\sigma_i = (\sigma_i^{\mathrm{p}}, \sigma_i^{\mathrm{r}})$ stationary if every agents strategy does not depend on history but soley on his position and the game in the current period. So, for agent we say a *stationary equilibrium* is a subgame perfect equilibrium in which each agent plays a stationary strategy. We call a *limit equilibrium* payoff vector the limit of the payoff vectors of stationary equilibria, i.e.

$$\lim_{d \to 0} \mathbf{u}(d).$$

[2]All results that follow can be easily adjusted to a more general specification $s\colon E \to \mathbb{R}^+$.

[3]However, it is easy to extend our game where s and p depend on t. We would denote this by a subscript t and assume knowledge of the network structure for all future periods.

As strategies are not allowed to depend on t, if p (or s) happen to depend t, we use the concept of belief-independent equilibrium in this case (especially in Proposition 10). Introducing the concept of belief-independent equilibria allows us to weaken the assumptions on perfect Information. A belief-independent equilibrium for a extensive game is a strategy profile that is optimal against every possible belief about the information set in this and all future periods. Or in other words for every profile of beliefs a strategy profile is called a belief-independent equilibrium if it is sequentially rational. In our case, beliefs are the equilibrium payoffs for all future rounds $(u_i^{t'})_{t' \geq t, i \in V}$.

We continue with a characterization of stationary equilibria by fixed point equations.

4 Characterising Stationary Models by Fixed-Point Equations

?, **?** characterized models of stationary bargaining with discounting to describe equilibrium outcomes. We do the same for the two cases of death rates.

Proposition 1 *All stationary equilibria of the unmatched leave model are payoff-equivalent. The equilibrium payoffs are exactly solutions to the fixed point equation*

$$u_i = \sum_{j \in V} p_{ij} \max\{s - u_j, u_i\} + (1 - \sum_{j \in V} p_{ij}) u_j - d \sum_{j,k \in V \setminus \{i\}} p_{jk} u_j$$

Here $p_{ij} = 0$ if $(i, j) \notin E$.

Proof Let agent i play σ - a stationary equilibrium of the game. As a result of stationarity, every agent i will have the continuation utility u_i. Now, suppose the game selects i to propose to j. Agent j will accept any offer above or equal her continuation utility u_j.
Given $u_i < s - u_j$ i will offer j exactly her continuation value u_j^{t+1} and j accepts. Any other offer above u_j would not be value-maximizing and therefore not part of an equilibrium. Analog, if $u_i > s - u_j$, any share that is acceptable for i will be rejected by j. Given equality agent i is indifferent between making the minimum offer or an unacceptable one, which does not make a difference for payoffs.
As this holds true for every link $(i, j) \in G$ we can derive following equilibrium equation

$$\begin{aligned} u_i &= \underbrace{\sum_{j \in V} p_{ij} \max\{s - u_j, u_i\}}_{i \text{ proposes}} + \underbrace{\sum_{j \in V} p_{ji} u_j}_{i \text{ receives}} + \underbrace{(1 - d) \sum_{j,k \in V \setminus \{i\}} p_{jk} u_j}_{i \text{ no trade}} \\ &= \sum_{j \in V} p_{ij} \max\{s - u_j, u_i\} + (1 - \sum_{j \in V} p_{ij}) u_j - d \sum_{j,k \in V \setminus \{i\}} p_{jk} u_j \end{aligned}$$

The equation is composed of following parts. Agent i either makes or receives an offer or stays unmatched and then either dies or not. Agent i is chosen to be the proposer with probability $\sum_{j\in V} p_{ij}$ and in line with the above argument receives either $s - u_j$ or u_i. If i receives a offer she gets her continuation utility for sure. In this setup the only possibility for i to die, with probability $d \in [0,1]$, is if a link is chosen that doesn't involve her.

Following similar reasoning, we can derive a similar equation for the *no agreement leave* specification:

Proposition 2 *All stationary equilibria of the no agreement leave model are payoff-equivalent. The equilibrium payoffs are exactly solutions to the fixed point equation*

$$u_i = \sum_{j\in V} p_{ij} \max\{s - u_j, u_i\} + \sum_{j\in V} p_{ji} u_j + \sum_{j,k\in V\setminus\{i\}} (1-d)u_i - d\sum_{j\in V}(p_{ji}+p_{ij})1_{u_i+u_j>s}u_i.$$

The only difference to the *unmatched leave* model is that in the case of no agreement both agents involved in the bargaining process could also die. This is captured by the last summand.

We are interested in limit equilibrium payoffs. In our case this means we want to see what happens if we let $d \to 0$, i.e. let the probability that agents leave the game without trading get infinitesimally small. As a next step we consider examples.

5 Examples

In the following three sections, we consider three example graphs. We contrast both specifications with **?** and conclude with remarks on the outcomes. We always assume $p_{ij} = \frac{1}{|E|}$, where $|E|$ is the number of links and $s = 1$.

5.1 One Monopolist

Consider Figure 1. It is well known that for the model of **?**, equilibrium payoffs

$$u_a = \frac{2}{3} \qquad\qquad u_b = u_c = \frac{1}{3}$$

The setup is as simple as it gets, we have one player a that is linked to both other players b and c who do not share a link with each other. Thus, in the proposed bargaining game a is bargaining every round for sure and the probability that she is proposing is $\frac{1}{2}$ while the others are the proposers of the bargaining game with $p_{ba} = \frac{1}{4}$ resp. $p_{ca} = \frac{1}{4}$. One could compare the position of a to the one of a monopolist. Our intuition would lead us to the conclustion that the

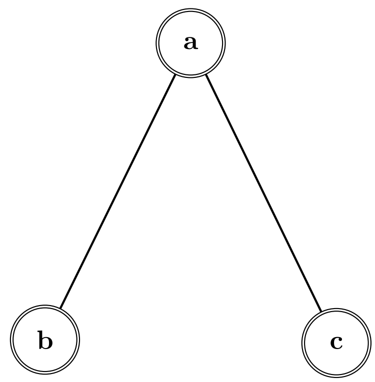

Figure 1: A first example: Monopolist a faces players b, c, which may not trade with each other.

monopolists exploits her market power to suppress the other players. However, in **?** the isolated players at the bottom have some power due to the endogenous inflow of new players. In contrast our example shows that our model with the *unmatched leave* specification is not in odds with the above intuition.

5.1.1 Unmatched Leave model

First, we consider the case in which only unmatched agents leave the game due to the death rate. Given this model we can, without any computations, make some relevant observations. The so-called monopolist a will be part of the bargaining game in every round and therefore is never threatened by the danger of dying. Her equilibrium equation takes the following form:

$$u_a = \sum_{j \in V} p_{aj} \max\{s - u_j, u_a\} + (1 - \sum_{j \in V} p_{aj}) u_a$$

As a next step we take a closer look at one of the two other agents. As the matching probabilities for b and c are symmetric, $u_b = u_c$ it suffices to consider b. This leads to

Proposition 3 $(1, 0, 0)$ *is the unique stationary equilibrium payoff vector for the unmatched leave model for any* $d > 0$*.*

Proof We have following equation for agent a:

$$u_a = \frac{1}{2} \max\{s - u_b, u_a\} + \frac{1}{2} u_a$$

Now for agent b:

$$u_b = \frac{1}{4}\max\{s - u_a, u_b\} + \frac{1}{4}u_b + (1-d)\frac{1}{2}u_b$$
$$= \frac{1}{4}\max\{s - u_a, u_b\} + \frac{3}{4}u_b - d\frac{1}{2}u_b$$

We have two different cases two consider. If $u_a > 1 - u_b$, then no trade happens, and we get the equations

$$u_a = u_a \qquad\qquad u_b = u_b - \frac{d}{2}u_b$$

meaning $u_b = 0$, which is a contradiction to $u_a > 1 - u_b$.

Otherwise, it must be the case that $u_a = 1 - u_b$ for the first equation to be true. We thus have the following condition $1 - u_b = u_a$. Plugging this into the equation for b we get again:

$$u_b = u_b - \frac{d}{2}u_b$$

Thus, unique payoff vector is $(a, b, c) = (1, 0, 0)$.

From the above argument we can see that the introduced death rate leads to the desirable allocation.
The force that causes this result is the fact that the introduced exogenous death rate for unmatched agents is irrelevant for agent a. As the agent will be part of the bargaining process in every period she never really faces the consequence of death. Thus, a has power over both other agents in so far that she can reject any offer below s without facing consequences. On the other side rejecting a offer by a ends with probability d in a payoff of 0.

This might be one of the weaknesses of the proposed *unmatched leave* stationary model. Although it feels natural that a person that is part of the bargaining process more often is less constrained by death, the games lead to some idiosyncrasies. This is due to the fact that an agent only properly discounts if she is with a sufficient high probability not matched.

5.1.2 No Agreement Leave model

The no-agreement leave model has the same unique payoff vector

Proposition 4 $(1, 0, 0)$ *is the unique stationary equilibrium payoff vector for the no agreement leave model for any* $d > 0$.

This merely follows from the following observation:

Proposition 5 *Consider a node* $|\delta^+(v)| = 1$ *in the unmatched leave or the no agreement leave model. Hence there is only one link leaving* v. *Let this link be* $e = (v, w)$. *Additionally, assume there is a link that neither ends in* v *nor starts there, which has positive probability. Then* (v, w) *is established, i.e.*

$$u_v + u_w \leq 1$$

Proof Assume not. Then the payoff of w must obey

$$u_v = \sum_{w \in V} (p_{vw} + p_{wv}) u_v + (1 - \sum_{w \in V} (p_{vw} + p_{wv})(1 - d) u_v$$

in the unmatched leave model. In the no-agreement model, the same holds with inequality $\leq$.

By assumption $\sum_{w \in V} (p_{vw} + p_{wv}) < 1$ and hence there is a constant $c < 1$ such that

$$u_v \leq c u_v$$

Yielding $u_v = 0$. But this means, that $u_v + u_w \leq 1$. This is a contradiction as we assumed that the link (v, w) was not established.

We thus know that all links will be established in an equilibrium for this example. But for the case that all links are established, the unmatched leave and the no agreement leave models are equivalent. Hence they also have the same payoff vectors. This last conclusion will not be true for more delicate examples.

5.2 An Example Similar to ?

Consider Figure 2. For this example,the model of **?** gives

$$u_b = u_x = u_c = \frac{1}{4} \qquad u_e = \frac{1}{2} \qquad u_d = \frac{1}{2} \qquad u_a = \frac{3}{4}$$

The following example adds 3 new players and links to the above discussed

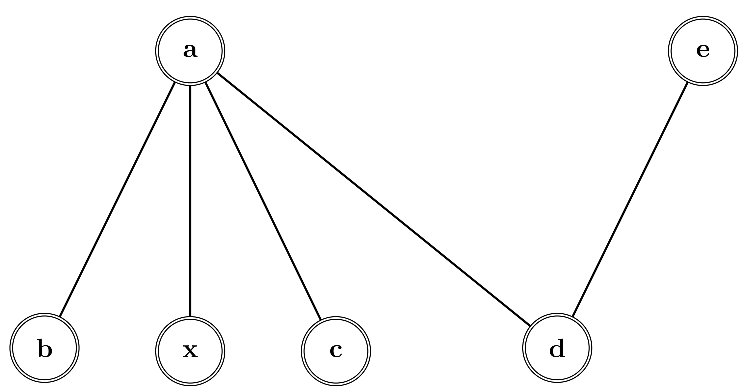

Figure 2: An example similar to one from **?**

network. **?** and **?** show that as their discount factor $\delta \to 1$ the link between a and d will vanish, this means that in the limit trade between a and d will happen with probability 0. The ratio behind this is as players become more patient they might find it favourable to bargain only with agents they have power over.

5.2.1 No Agreement Leave Model

Proposition 6 *For the above example, there is a unique equilibrium* $(a, b, c, x, d, e) = (\frac{6}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{2}, \frac{1}{2})$.

Proof We make a case distinction on whether the link (a, d) is established. If it is not established, then the equations for e and d are symmetric with recursive equation

$$u_e = \frac{1}{10}\,(1 - u_d) + \frac{1}{10}u_e + \frac{4}{5}(1 - d)u_e$$

In the no agreement leave model, by symmetry, $u_e = u_a$. Having this, we arrive by function manipulation at

$$u_d = u_e = \frac{1}{10(\frac{1}{5} + \frac{4}{5}d)}.$$

By symmetry, $u_b = u_c = u_x$. Therefore, it suffices to only consider u_b. We arrive at the equation system

$$u_a = \frac{3}{10}(1 - u_b) + \frac{3}{10}u_a + \frac{4}{10}u_a$$
$$u_b = \frac{1}{10}(1 - u_a) + \frac{1}{10}u_b + \frac{8}{10}(1 - d)u_b$$

which has the following solution:

$$u_a = \frac{6}{7 + 8d} \qquad\qquad u_b = u_x = u_c = \frac{1}{7 + 8d}$$

Which converges to the equilibrium $(a, b, c, x, d, e) = (\frac{6}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{2}, \frac{1}{2})$.

Now we consider the case when (a, d) is established. Then we have the following system of equations:

$$u_b = \frac{1}{10}\,(1 - u_a) + \frac{1}{10}u_b + \frac{4}{5}(1 - d)u_b$$
$$u_e = \frac{1}{10}\,(1 - u_d) + \frac{1}{10}u_e + \frac{4}{5}(1 - d)u_e$$
$$u_a = \frac{1}{5}(1 - u_b) + \frac{1}{10}(1 - u_d) + \frac{3}{10}u_a + \frac{2}{5}(1 - d)u_a$$
$$u_d = \frac{1}{10}(1 - u_a) + \frac{1}{10}(1 - u_e) + \frac{1}{5}u_d + \frac{3}{5}(1 - d)u_d$$

This results in

$$u_a = \frac{2\left(72d^2 + 19d\right)}{192d^3 + 232d^2 + 54d + 1}$$
$$u_b = u_c = \frac{24d^2 + 8d + 1}{192d^3 + 232d^2 + 54d + 1}$$
$$u_d = \frac{64d^2 + 32d + 1}{192d^3 + 232d^2 + 54d + 1}$$
$$u_e = \frac{2\left(96d^3 + 84d^2 + 11d\right)}{1536d^4 + 2048d^3 + 664d^2 + 62d + 1}$$

This would converge to the payoff vector $(a, b, c, x, d, e) = (0, 1, 1, 1, 0)$, but this is not an equilibrium. As for small enough d,

$$u_a + u_d > 1$$

which means, that in contradiction to our assumption, no link between a and d is established. Thus, there is no equilibrium for any small enough d.

There is no equilibrium more specifically for $d < \frac{1}{48}\left(\sqrt{201} - 3\right) \approx 0.23$.

5.2.2 Unmatched Leave Model

In the first part of this section we already established that all isolated players will trade for sure - the argument for the umatched leave model is similar.

Proposition 7 *For the above example, there are two equilibria in the unmatched leave model:* $(a, b, x, c, d, e) = (\frac{12}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{4}{7}, \frac{3}{7})$ *and* $(1, 0, 0, 0, 1)$.

Proof Again we will make a case distinction. First consider the case that link (d,e) is not established. This means that a and d will not trade with each other. The set of equations is

$$\begin{aligned} u_a &= \frac{3(1-u_b)}{3+2d} \\ u_b = u_x = u_c &= \frac{1-u_a}{1+8d} \\ u_d &= \frac{(1-u_e)}{1+6d} \\ u_e &= \frac{1-u_d}{1+8d} \end{aligned}$$

Solving above equations gives:

$$\begin{aligned} u_a &= \frac{12}{13+8d} \\ u_b = u_x = u_c &= \frac{1+8d}{64d^2+112d+13} \\ u_d &= \frac{4}{7+24d} \\ u_e &= \frac{3+24d}{192d^2+80d+7} \end{aligned}$$

For $d \to 0$ this converges to $(a, b, x, c, d, e) = (\frac{12}{13}, \frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{4}{7}, \frac{3}{7})$. For no trade in the link between a and d we further need: $u_a + u_d > 1 = s$ which is given for any $d < \frac{3}{8}$. All other inequalities for the establishment of links are satisfied, as

one checks.
If the link is established we have following system of equations:

$$
\begin{aligned}
u_a &= \frac{4-(3u_b+u_d)}{4+2d}\\
u_b = u_x = u_c &= \frac{1-u_a}{1+8d}\\
u_d &= \frac{2-(u_e+u_a)}{2+6d}\\
u_e &= \frac{1-u_d}{1+8d}
\end{aligned}
$$

Solving this set of equations gives:

$$
\begin{aligned}
u_a &= \frac{768d^3+440d^2+39d+1}{384d^4+992d^3+438d^2+36d+1}\\
u_b = u_x = u_c &= \frac{48d^3+22d^2-3d}{384d^4+992d^3+438d^2+36d+1}\\
u_d &= \frac{128d^3+152d^2+5d}{384d^4+992d^3+438d^2+36d+1}\\
u_e &= \frac{48d^3+102d^2+23d+1}{384d^4+992d^3+438d^2+36d+1}
\end{aligned}
$$

For $d \to 0$, this converges to $(1,0,0,0,1)$. All inequalities for the establishment of links are satisfied.

This means, that in both cases, we have a a more unequal distribution of utility than in the model of Menea.

Last, we consider one of the smallest examples with a cycle.

5.3 An Example with a Cycle

Consider Figure 3. This last example contains a cycle. **?**'s payoffs are

$$
u_b = u_c = u_e = \frac{4}{10} \qquad\qquad u_a = u_d = \frac{6}{10}
$$

Because of symmetry, we may assume that b and c get the same payoffs.

5.3.1 No-Agreement leave model

Proposition 8 *The above example has the unique equilibrium* $(a,b,c,d,e) = (\frac{8}{11}, \frac{3}{11}, \frac{3}{11}, \frac{1}{2}, \frac{1}{2})$.

Proof We have to consider four cases:

1. Neither of the edges are established: This is clearly impossible, as isolated edges get zero payoff and hence all edges should be established.

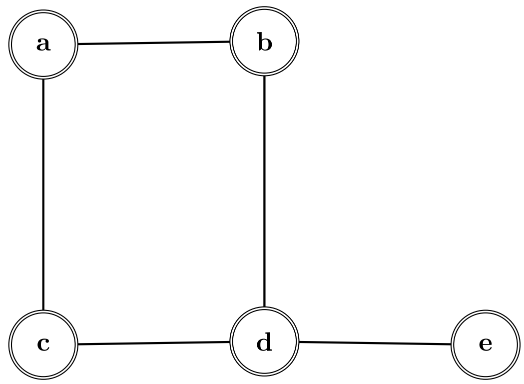

Figure 3: An example with a cycle.

2. The edges (a, b) and (a, c) are not established and the ones between (c, d), (b, d) are. Then player a is isolated. Hence she gets zero payoff and we arrive again at a contradiction.

3. The last remaining case is the case that all edges are established. Then we arrive at a linear equation system (where we only need to specify the payoffs for a, b, d, e, as necessarily $u_b = u_c$. Then

$$\begin{aligned} u_a &= \frac{1}{5}(1-u_1) + \frac{1}{5}u_2 + \frac{3}{5}(1-d)u_2 \\ u_b = u_c &= \frac{1}{10}(1-u_2) + \frac{1}{10}(1-u_3) + \frac{1}{5}u_1 + \frac{3}{5}(1-d)u_1 \\ u_d &= \frac{1}{5}(1-u_1) + \frac{1}{10}(1-u_4) + \frac{3}{10}\ u_3\ + \frac{2}{5}\ (1-d)u_3 \\ u_e &= \frac{1}{10}(1-u_3) + \frac{1}{10}u_4 + \frac{4}{5}(1-d)u_4 \end{aligned}$$

Solving, we arrive at the outcomes

$$\begin{aligned} u_a &= \frac{192d^3 + 200d^2 + 40d + 1}{2d\,(288d^3 + 444d^2 + 178d + 15)} \\ u_c = u_b &= \frac{96d^3 + 36d^2 - 13d - 1}{2d\,(288d^3 + 444d^2 + 178d + 15)} \\ u_d &= \frac{288d^3 + 162d^2 + 22d + 1}{2d\,(288d^3 + 444d^2 + 178d + 15)} \\ u_e &= \frac{72d^3 + 66d^2 + 16d - 1}{2d\,(288d^3 + 444d^2 + 178d + 15)} \end{aligned}$$

for which for small enough d, $u_d + u_e > 1$.

4. The edges (a, b) and (a, c) are established and the ones between (c, d), (b, d) not. This case is similar to the one for which we established an

equilibrium for the last example. For example,

$$u_d = u_e = \frac{1}{10(\frac{1}{5} + \frac{4}{5}d)}.$$

Furthermore, again b and c get the same payoff by symmetry. Therefore, we arrive at the equation system

$$\begin{aligned} u_a &= \frac{2}{10}(1 - u_b) + \frac{2}{10}u_a + \frac{6}{10}u_a \\ u_b &= \frac{1}{10}(1 - u_a) + \frac{1}{10}u_b + \frac{8}{10}(1 - d)u_b \end{aligned}$$

which has the solution

$$u_a = \frac{8}{11 + 24d} \qquad u_b = u_c = \frac{3}{11 + 24d}$$

which converges to the payoff vector $(a, b, c, d, e) = (\frac{8}{11}, \frac{3}{11}, \frac{3}{11}, \frac{1}{2}, \frac{1}{2})$.

5.3.2 Unmatched Leave model

Proposition 9 *In the unmatched leave model, there is a unique equilibrium of* $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3})$.

Proof Here we arrive at the system of equations

$$\begin{aligned} u_a &= \frac{2 - 2u_b}{2 + 6d} \\ u_b = u_c &= \frac{2 - (u_a + u_d)}{2 + 6d} \\ u_d &= \frac{3 - (2u_b + u_e)}{3 + 4d} \\ u_e &= \frac{1 - u_d}{1 + 8d} \end{aligned}$$

Solving gives:

$$\begin{aligned} u_a &= \frac{96d^2 + 84d + 10}{288d^3 + 444d^2 + 178d + 15} \\ u_b = u_c &= \frac{96d^2 + 64d + 5}{288d^3 + 444d^2 + 178d + 15} \\ u_d &= \frac{2(108d^2 + 57d + 5)}{288d^3 + 444d^2 + 178d + 15} \\ u_e &= \frac{36d^2 + 24d + 5}{288d^3 + 444d^2 + 178d + 15} \end{aligned}$$

For $d \to 0$ the payoffs converge to $(a, b, c, d, e) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3})$.

Here again, we notice that the payoffs are more unequal for models with death than for the discounting model.

5.4 Concluding Remarks

We showed that equilibrium outcomes structurally resemble the outcomes of the model of **?**, although the distribution is generally more unequal. It can be claimed, that equilibriums exist and are unique for the *no agreement leave* model. These results already show, that the *unmatched leave* model does not always admit a unique solution. For the latter, we are going to study other types of added friction to establish equilibrium existence and uniqueness in the next section.

6 Equilibrium Existence and Uniqueness by Adding Small Discounting

We present now how added friction regularizes the problem such that equilibrium existence and uniqueness hold, even for the *unmatched leave* model. Two alternatives are imaginable: First, in addition to having a death rate d, one could assume discounting $\delta < 1$ of any continuation payoffs in the last model. Then, if $(1, 0, 0)$ happens to be a robust outcome, when considering the equilibrium outcome vector

$$\mathbf{u}(\delta, d)$$

one should expect that the limit (where a vector of outcomes converges iff all components converge)

$$\lim_{d \to 0} \lim_{\delta \to 1} \mathbf{u}(\delta, d) \tag{1}$$

equals $(1, 0, 0)$.

Another possibility of additional friction is to, in each round, first decide to not choose any edge with a probability ϵ and otherwise to choose an edge and continue with the game as specified before yielding $\mathbf{u}(\epsilon, d)$ and then consider

$$\lim_{d \to 0} \lim_{\delta \to 1} \mathbf{u}(\epsilon, d).$$

We leave the second specification for further work. Numerical simulations show that equilibrium uniqueness is unlikely for this model.

Generalising the recursions from before to cases with time-dependent matching probabilities, one might claim, that

$$u_i^t = \sum_{j \in V} p_{ij}^t \max\{s - \delta u_j^{t+1}, \delta u_i^{t+1}\} + \sum_{j \in V} p_{ji}^t \delta u_j^{t+1} + (1-d) \sum_{j,k \in V \setminus \{i\}} p_{jk}^t \delta u_j^{t+1}$$

for the unmatched leave model and

$$\begin{aligned} u_i^t = & \sum_{j \in V} p_{ij}^t \max\{s - \delta u_j^{t+1}, \delta u_i^{t+1}\} + \sum_{j \in V} p_{ji}^t \delta u_j^{t+1} \\ & + \sum_{j,k \in V \setminus \{i\}} (1-d) \delta u_i^{t+1} + \sum_{j \in V} (1-d)(p_{ji}^t + p_{ij}^t) 1_{\delta u_i^{t+1} + \delta u_j^{t+1} > s} \delta u_i^{t+1} \end{aligned}$$

for the no agreement leave model should hold. The following shows for all $d \geq 0, \delta > 0$ equilibrium existence.

Proposition 10 *The recursion equation for the unmatched leave model has a unique fixed point, which is a belief-independent equilibrium.*

Note that here the fixed point equation is of vectors in the space $[0,1]^{E\times\mathbb{N}}$. We use a method as in **?**. Please see Theorem 2 there, which we generalise here.

PROOF We characterise belief-independent equilibria by repeatedly deleting conditionally dominated payoff values i.e. values that are either too high or too low to be supported by the continuation values in the next round. More specifically, starting with the highest and lowest beliefs an agent might have about equilibrium payoffs, i.e. $l_i^{t,0} = 0$ and $h_i^{t,0} = s$, $i \in V$,$t \in \mathbb{N}$, we give bounds $l_j^{t,k}$ and above $h_j^{t,k}$ for the expected payoff vectors that can be supported by payoff values within $[l_j^{t+1,k-1}, h_j^{t+1,k-1}]$. By showing, that the bound vectors $\mathbf{h}^k$ and $\mathbf{l}^k$ converge in $\|\cdot\|_\infty$-norm against each other, we establish equilibrium uniqueness

We define $l_j^{t,k}$, $h_j^{t,k}$ recursively. For $k = 0$, all strategies leading to payoffs outside of the interval $[l_j^{t,0} = 0, h_j^t = s], j \in N$ arise from strategies that yield a profitable deviation for any partner: Refuting all proposals yields these non-negative payoffs, which is higher than the payoff given here. Now define the two sequencies as the following (for fixed t). For the recursive step, we get

$$\begin{aligned} l_i^{t,k+1} &= \sum_{j\in V} p_{ij}^t \max\{s - \delta h_j^{t+1,k}, \delta l_i^{t+1,k}\} + \sum_{j\in V} p_{ji}^t \delta l_j^{t+1,k} \\ &\qquad + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t (1-d)\delta l_i^{t+1k} \\ h_i^{t,k+1} &= \sum_{j\in V} p_{ij}^t \max\{s - \delta l_j^{t+1,k}, \delta h_i^{t+1,k}\} + \sum_{j\in V} p_{ji}^t \delta h_j^{t+1,k} \\ &\qquad + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t (1-d)\delta h_i^{t+1k} \end{aligned}$$

Now consider a strategy, in which player i receives a payoff that is larger or equal than $h_j^{t,k+1}$. Then, we know by the induction hypothesis that any agent j will refute a proposal that yields her payoff lower than $l_j^{t+1,k}$. Thus, if i offers him less, j will have a profatible deviation. Furthermore, agent i has a continuation payoff of at most $h_i^{t+1,k}$, which he receives in case of being responder and with probability $1-d$ in case he is not matched. The lower bound is explained analogously.

Now we would like to bound the quantity $\Delta^k = \sup_{i\in V,t\in\mathbb{N}} h_i^{k,t} - l_i^{k,t}$ which is nonnegative by definition of the bounds. In this moment is becomes clear what δ changes for the equilibrium utilities. Consider

$$\begin{aligned}
\Delta^{k+1} = & \sup_{i\in V, t\in\mathbb{N}} \sum_{j\in V} p_{ij}^t(\max\{s-\delta l_j^{t+1,k}, \delta h_i^{t+1,k}\} - \max\{s-\delta h_j^{t+1,k}, \delta l_i^{t+1,k}\}) \\
& + \sum_{j\in V} p_{ji}^t(\delta h_j^{t+1k} - \delta l_j^{t+1k}) + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t (1-d)(\delta h_i^{t+1k} - \delta l_i^{t+1k}) \\
& \leq \sup_{i\in V, t\in\mathbb{N}} \sum_{j\in V} p_{ij}^t \max\{\delta h_j^{t+1,k} - \delta l_j^{t+1,k}, \delta h_i^{t+1,k} - \delta l_i^{t+1,k}\} \\
& + \sum_{j\in V} p_{ji}^t(\delta h_j^{t+1k} - \delta l_j^{t+1k}) + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t (1-d)(\delta h_i^{t+1k} - \delta l_i^{t+1k}) \\
& \leq \sup_{i\in V, t\in\mathbb{N}} \sum_{j\in V} p_{ij}^t \max\{\delta h_j^{t+1,k} - \delta l_j^{t+1,k}, \delta h_i^{t+1,k} - \delta l_i^{t+1,k}\} \\
& + \sum_{j\in V} p_{ji}^t(\delta h_j^{t+1k} - \delta l_j^{t+1k}) + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t (\delta h_i^{t+1k} - \delta l_i^{t+1k}) \\
& \leq \delta \sup_{i\in V, t\in\mathbb{N}} \sum_{j\in V} p_{ij}^t \max\{\Delta^k, \Delta^k\} + \sum_{j\in V} p_{ji}^t \Delta^k + \sum_{j,k\in V\setminus\{i\}} p_{jk}^t \Delta^k \\
& = \delta\Delta^k
\end{aligned}$$

Here, the first inequality uses the fact, that for any four real numbers a, b, c, d we have

$$\max\{a, b\} - \max\{c, d\} \leq \max\{a-b, c-d\}$$

The second uses $d \geq 0$ and $\mathbf{h}^k \geq \mathbf{l}^k$ componentwise. The third follows by the very definition of Δ^k. This establishes the claim.

This theorem allows efficient approximation of equilibrium outcomes. See accompanying code `recursion.py`.

It is not straightforward to prove a similar result for the no agreement leave model: There, the indicator functions $1_{u_i+u_j>s}$ may be different for upper and lower bounds. Differently phrased, upper and lower bounds on non-dominated payoffs could diverge in the links that are established, and by having many edges in a lower bound, that are established, the quantity Δ^k might be increasing.

7 Conclusions and Further Work

We showed variants of stationary bargaining models with an exogenous death rate. We calculate numerical examples and show that equilibrium uniqueness is not given for one of the variants. For this variant of the models we show that by adding discounting, this uniqueness and existence can be shown.

Open questions remain: Do equilibria always exist for the two presented models? Are these always unique for the no agreement leave model? In the unmatched leave model: Which edge sets can be present in some equilibria an in others not?

Having uniqueness and existence of equilibrium payoff vectors and continuity, we show in an appendix how this can be used to infer equilibrium existence in a related dynamic model.

A From Stationary to Dynamic

We now show how results like Proposition 10, may be used to analyse models with exogenous inflow, which generalise results from **?**. We first give a dynamic model, for which we can show equilibrium existence using what we showed before.

A.1 A Model of Exogenous Inflow and a Continuum of Agents

We describe a model of dynamic bargaining in the spirit of **?**, which resembles a model of **?**. We fix again a graph G. This time, the matching probabilities $\mathbf{p}$ are not fixed. Instead, we fix *inflow*: At each $t \in \mathbb{N}$ (call this *in each round*) there is a continuum of agents at each vertex $i \in V$. We call these agents *of type* i. To the continuum at $i \in V$, at the beginning of each round, a mass of agents of measure $\lambda_i^t \in \mathbb{R}^+$ is added, the *exogenous inflow*. We define the sequence $(\mu_i^t)_{t\in\mathbb{N}}$, the stock of agents of type i present in round t recursively: $\mu_i^0 = \lambda_i^0$, $\forall i \in V$, i.e. the initial stock of agents is given by the exogenous inflow in round 0. At each t, a_e^t is the endogenous *agreement rate* will denote the agreement rate, which will be defined later. Then, we define

$$\mu_i^{t+1} := \mu_i^t - \sum_{e\in\delta(i)} a_e^t \beta_e(\mu^t) - d(1 - \sum_{e\in\delta(i)} \beta(\mu^t)),$$

which we call the *unmatched leave* model (as in the exogenous matching probability case). For the *no agreement leave model*, one substracts $d\sum_{e\in\delta(i)} \beta_e(\mu^t)(1-a_e^t)$. Here, β denotes the linear matching technology[4]

$$\beta\colon \mathbb{R}_+^V \to \mathbb{R}_+^E, \beta_{e=(i,j)}(\mu^t) = \overline{p}_e \mu_i^t \mu_j^t,$$

which denotes for $e = (i, j)$ the measure of the random set of agents of type i that propose to agents of type j[5]. Here, $\overline{p} : E \to [0, 1]$ is a probability mass function. Note the difference of $\frac{1}{2}$ to the definition in e.g. **?**. This emerges from our definition of the graph as directed (we split a bidirected edge in two directed, contrary edges with half as high probability). As all agents of type i are equally likely to be proposer, an agent of type i proposes to then with a probability of

$$p_{e=(i,j)}(\mu^t) = \overline{p}_e \mu_j^t$$

[4]We assume this for ease of exposition. In the following we merely use continuity and *normalization* of the matching technology, i.e. $\sum_{e\in E} \beta_e(\mu^t) = 1$ for any μ_t.

[5]This can be done in a way that all sets are measurable.

It remains the definition of agreement rates $a^t_{e=(i,j)}$ which comes from our dynamics: It is the ratio of agents of type i that propose successfully to j in round t and leave the game. Assuming fixed matching probabilities p, we can use the recursion equations established in the last sections to define a function (because of uniqueness)

$$\mathbf{p} \mapsto \mathbf{u}$$

which is continuous in the $\|\cdot\|$-norm in consequence of Proposition 10 (One can use the proof of Theorem 2 (v) of **?** verbatim). Finally, **?** uses the Kakutani-Glicksberg fixed point theorem to establish equilibrium existence by showing that fixed points of the correspondence

$$\mathbf{a} \mapsto \boldsymbol{\mu} \mapsto \mathbf{p} \mapsto \mathbf{u} \rightrightarrows \mathbf{a}$$

correspond to belief-independent equilibria.

All of the functions in this compositions are after continuous after inspection of their respective definition. The last correspondence is closed and convex valued (either a player is indifferent whether to propose or not, then the image is $[0, 1]$) or it is $\{0\}$ resp. $\{1\}$. Therefore, one can "lift" result for a stationary model to dynamic models for all cases in which $\mathbf{p} \mapsto \mathbf{u}$ is continuous.