Two Technical Results on Pareto Frontiers

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We give an alternative and shorter proof of a characterisation result of Pareto frontier in Mennle and Seuken 2016 and present a unified language, multicriteria optimisation, to talk about Pareto frontiers in the generality in which their results hold.

We formalise a concept of convergence of Pareto frontiers and give an alternative formulation of one of the main theorems in Tatur 2005.

1. Introduction

Famous impossibility results in mechanism design and social choice show that some desiderata for mechanisms cannot be satisfied simultaneously in private value settings. One example of this is the Myerson-Satterthwaite theorem (Myerson and Satterthwaite 1983). It shows that the desiderata interim efficiency and budget balance are incompatible in a Bayesian bilateral trade setting. These desiderata can therefore not be used as criteria for selection of mechanism to be used in practice.

If one were able, however, to quantify by how much both budget balance and efficiency are violated, one could consider optimal payoffs between the two. This is possible for a wide range of models. Consider for example the mechanism design problem of allocating stochastically a single good to n sellers: If the payments of sellers $i = 1, \dots, n$ with types $\theta = (\theta_1, \dots, \theta_n)$ are denoted $t_i(\theta)$ and the allocation probabilities by $q_i(\theta)$, then a violation of budget balance, the *deficit*, can be measured by $\max\{0, \mathbb{E}[\sum_{i=1}^n t_i(\theta)]\}$. Inefficiency can also be quantified: If q_i is the (stochastic) allocation of a single good to agent $i = 1, \dots, n$, then $\mathbb{E}[\max_{i=1,\dots,N} \theta_i]$, the expected total valuation in the optimal allocation to the agent having the highest valuation for the good, net of $\mathbb{E}[\sum_{i=1}^N \theta_i q_i]$, the expected valuation of the agents in the mechanism, can be used to measure inefficiency.

One would like to minimise *simultaneously* these two functions subject to individual rationality (IR) and incentive compatibility (IC) constraints. If no prior information on how to weigh the two is given, the only conclusions one can draw is on Pareto optima. This note presents results of Mennle and Seuken 2016 in a language of *multicriteria optimisation* and formalises convergence of Pareto frontiers.

The plan of the note is as follows: In section 2 we present general results on bicriteria optimisation and how these imply the results in Mennle and Seuken 2016. In section 3 we consider limits of Pareto frontiers and give another formulation of a major result in Tatur 2005 can be phrased. Proofs are in Appendix A.

2. Characterising Pareto Frontiers

Consider a mechanism design problem. Let Ω be a set of mechamisms (e.g. those that are ex ante IR and interim IC) and define functions f_i such as the *deficit* or the *inefficiency* above. We will call those functions occasionally *desiderata*. One notices that these have a certain structure: Ω will most likely be a convex subset of a vector space, the both functions mentioned above are linear[1]. We define convex and polyhedral multicriteria optimisation problems as objects that capture this structure.

2.1. Pareto Frontiers of Multicriteria Optimisation Problems

We first define the central objects of this note.

Definition 2.1. 1. A *k-criteria optimisation problem* is a tuple

$$P = (f_1, \dots, f_k, \Omega),$$

where $f_i \colon \Omega \to \mathbb{R}$. For such a problem, we denote $f \colon \Omega \to \mathbb{R}^k, x \mapsto (f_1(x), \dots, f_k(x))$.

2. The *solution* to a k-criteria optimisation problem is the set of pareto minima of $f_1, \dots, f_k$, i.e.

$$\Omega \supseteq \Omega_{(f_1,\dots,f_n)} := \{x \in \Omega | \nexists y \colon \forall i = 1, \dots, k \colon f_i(y) \leq f_i(x) \\ \text{and } \exists j \in \{1, \dots, k\} \colon f_i(y) < f_i(x)\}.$$

3. Then the *feasible set* is

$$F_{(f_1,\dots,f_k,\Omega)} := f(\Omega) + \mathbb{R}^k_+ \ \subseteq \mathbb{R}^k$$

4. The *Pareto frontier* is

$$P_{(f_1,\dots,f_k,\Omega)} := f(\Omega_{(f_1,\dots,f_n)}) \ \subseteq \mathbb{R}^k$$

Equivalently we could have defined the Pareto frontier as the set of Pareto minimal points in the feasible set.

In most finite type settings, the set of feasible mechanisms is a polyhedron:

[1] We will make the sense of linearity clear in an example in (2.2)

Definition 2.2 (Polyhedron)**.** A set A is a *polyhedron* if there are $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ such that
$$A = \{\mathbf{x} \in \mathbb{R}^n | \ \mathbf{A}\mathbf{x} \geq \mathbf{b}\}.$$

We call an (k-criteria) optimisation problem $(f_1, \ldots, f_k, \Omega)$ *convex* if all $f_i, i = 1, \ldots, k$ are convex and the set of feasible points Ω is a convex subset of a vector space V. We call an optimisation problem *polyhedral* if in addition Ω is a polyhedron and all $f_i, i = 1, \ldots, k$ are defined as the maximum over a finite number of linear functions. The main result of this note is the following:

Theorem 2.3.
1. *Let $(f_1, \ldots, f_k, \Omega)$ be a convex k-criteria optimisation problem. Then $F_{(f_1,\ldots,f_k,\Omega)}$ is a convex set. In addition, $F_{(f_1,\ldots,f_k,\Omega)}$ is a graph of a convex function $f\colon \mathbb{R}^{k-1} \to \mathbb{R}$.*
2. *Let $(f_1, \ldots, f_k)$ be a polyhedral k-criteria optimisation problem. Then $F_{(f_1,\ldots,f_k,\Omega)}$ is a polyhedron.*
3. *If in addition $f_i(\Omega) \subseteq \mathbb{R}_+$, then*
$$F_{(f_1,\ldots,f_k,\Omega)} = \mathrm{conv}(\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}) + \mathbb{R}^k_+ \subseteq \mathbb{R}^k.$$
4. *In particular, if in addition $k = 2$, then for some points $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^k$*
$$P_{(f_1,\ldots,f_k,\Omega)} = \bigcup_{i=1}^{m-1} \mathrm{conv}(\{\mathbf{x}_i, \mathbf{x}_{i+1}\}).$$

In virtually all finite type mechanism design settings, the Pareto frontier of most reasonable desiderata will be the solution to a polyhedral k-criteria optimisation problem. In continuous type mechanism design settings, it will in most cases be the solution to a convex k-criteria optimisation problem. Mennle and Seuken 2016 showed item 4 of this theorem in a much more concrete setting. We will describe this setting in subsection 2.2.

Another question is when one can consider w.l.o.g. mechanisms that have an invariance property under some kind of group action. Two examples of this (from social choice) are the permutation of agents — the corresponding invariance property is *anonymity* — and the permutation of outcome options — the corresponding invariance property is *neutrality*. But one could also think of other settings, e.g. invariance under permutations within two groups, such as permutations of sellers and permutation of buyers with swapping a buyer and a seller not being allowed. Such an invariance corresponds to symmetric equilibria of the games corresponding to mechanisms and is therefore often (e.g. in Tatur 2005) called *symmetry*. The following shows, that the question raised at the beginning of this paragraph can be answered with *yes* in many settings.

Proposition 2.4. *Let $(f_1, \dots, f_k, \Omega)$ be a convex k-criteria optimisation problem. Let $G \times \Omega \to \Omega, (g, x) \mapsto g.x$ be a group action[2] such that*

$$f_i(g.x) = f_i(x), \qquad \forall x \in \Omega, i = 1, \dots, k$$

Then for any $\mathbf{x} \in P_{(f_1,\dots,f_k,\Omega)}$, there is a $y \in f^{-1}(\{\mathbf{x}\})$ that is invariant under G's action, i.e. $g.y = y, \forall g \in G$.

For the special case of anonymity, this result means, that if all functions in a multicriteria optimisation problem are anonymous (in the sense of invariant under permutation of agents), then for each point on the Pareto frontier one finds an anonymous mechanism that is mapped to this point by f.

In the rest of this section, we show that most results in Mennle and Seuken 2016 are special cases of the results in this section.

2.2. Reproving major results in Mennle and Seuken 2016

Mennle and Seuken 2016 consider a randomised social choice setting. More specifically, let $N = \{1, \dots, n\}$ be players and $M = \{1, \dots, m\}$ be objects. Let player i have type (P_i, u), where $P_i \in \mathcal{P}(N)$ is a preference order on the elements of N and u is a utility function $u \colon M \to [0, 1]$[3] such that $i \preccurlyeq_{P_i} j \Rightarrow u(i) \leq u(j)$. Suppose players simultaneously send reports m_i to the principal and the object is chosen according to a probability distribution with mass function $\mathbf{q}(m_1, \dots, m_n) \in \Delta(M)$. We denote $\mathbf{q}$'s coordinates by $q_i(m_1, \dots, m_n)$, $i \in M$. Thanks to the revelation principle, we can equivalently consider the set of direct mechanisms, $\mathcal{M}$, which consists of functions

$$\varphi \colon (\mathcal{P}(M))^n \to \Delta(M)$$

that are IR, i.e.

$$\sum_{i=1}^{m} u_i(m)\varphi_m(P_1, \dots, P_n) \geq 0,$$
$$\forall i \in N, \forall (P_1, \dots, P_n) \in (\mathcal{P}(M))^n, \forall (u_1, \dots, u_n) \in U_{P_1} \times \cdots \times U_{P_n}. \quad (1)$$

and IC, i.e.

$$\sum_{i=1}^{m} u_i(m)(\varphi(P_i', P_{-i}) - \varphi(P_1, \dots, P_n)) \leq 0,$$
$$\forall i \in N \forall (P_1, \dots, P_n) \in (\mathcal{P}(M))^n, P_i' \in \mathcal{P}(M), (u_1, \dots, u_n) \in U_{P_1} \times \cdots \times U_{P_n}. \quad (2)$$

[2]That is, $e.x$ for the identity e of G and $g.(h.x) = (gh).x$ for any $x \in \Omega$ and $g, h \in G$.

[3]The restricted range (as opposed to $\mathbb{R}_+$ will not change the set of feasible mechanisms, as both IC and IR constraints are invariant under multiplication with positive constants. For a relaxation of ICconstraints as the one presented below, however, this will play a role.

where we write $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$, for $i = 1, \ldots, n$. We denote the set of mechanisms that are merely IR and not necessarily IC by $\mathcal{M}_{\mathrm{IR}}$.

We can form linear combinations of mechanisms by defining for $\varphi, \psi \in \mathcal{M}$

$$(\alpha\varphi + \psi)(P_1, \ldots, P_n) \coloneqq \alpha\varphi(P_1, \ldots, P_n) + \psi(P_1, \ldots, P_n)$$

If we take convex combinations of two mechanisms, $\lambda\varphi + (1-\lambda)\psi$, this corresponds to the mechanism that one gets by initially for each reports deciding with probability λ to apply mechanism φ and with probability $1-\lambda$ to apply mechanism ψ. This gives $\mathcal{M}$ and $\mathcal{M}_{\mathrm{IR}}$ the structure of convex subsets of vector spaces. As constraints (1) and (2) are linear, $\mathcal{M}$ and $\mathcal{M}_{\mathrm{IR}}$ can be written in the form

$$\{\mathbf{x} \in \mathbb{R}^{\ell} |\ \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$$

for some $\ell \in \mathbb{N}$. Hence, $\mathcal{M}$ and $\mathcal{M}_{\mathrm{IR}}$ are polyhedra.

Now consider any two functions $f, g : \mathcal{M} \to \mathbb{R}_+$ that are the maximum of finitely many linear functions and such that $0 \in f(\mathcal{M})$ and $0 \in g(\mathcal{M})$[4]. One example of such a function is:

$$f(\varphi) = \max \left\{ \max_{\substack{(P_1,\ldots,P_n) \in (\mathcal{P}(M))^n \\ P'_i \in \mathcal{P}(M) \\ (u_1,\ldots,u_n) \in \hat{U}_{P_1} \times \cdots \times \hat{U}_{P_n}}} \sum_{i \in N} \sum_{i=1}^{m} u_i(m)(\varphi(P'_i, P_{-i}) - \varphi(P_1, \ldots, P_n)), 0 \right\} \tag{3}$$

where $\hat{U}_i = \{u \colon M \to \{0,1\} \mid i \preccurlyeq_{P_i} j \Rightarrow u(i) \leq u(j)\}$ (note $|U| = k$). This looks complicated, but to apply the results in this note, one only has to observe that it is the maximum over finitely many linear functions over mechanisms. For $\mathcal{M}$, this desideratum is trivial, $f(\mathcal{M}) = \{0\}$, as (1) implies that the inner maximum is bounded above by 0 and taking the maximum with 0 implies that this function constantly evaluates to 0. We will come back to the meaning of this function in a remark below.

Let $n_i(P_1, \ldots, P_n), i \in M$ be the number of times $i \in M$ is a first choice in $P_1, \ldots, P_n$. Then another function that is linear in φ is

$$g(\varphi) = \max \left\{ \max_{P=(P_1,\ldots,P_n) \in \mathcal{P}(M)^N} \left(\left(\max_{j \in M}\ n_j(P) \right) - \sum_{j \in M} \varphi(P)_j n_j(P) \right), 0 \right\}$$

Note that $\max_{j \in M}\ n_j(P)$ is a constant. Therefore, this function is again a maximum over linear functions of mechanisms, which means that the results from above are applicable. This function is defined in Mennle and Seuken 2016 as *plurality*, e.g. how much is the deviation from the majority choice in the worst case.

[4]This ensures that points $(0, y)$, $(x, 0)$ are in the Pareto frontier for some $x, y \in \mathbb{R}$.

Consider the 2-criteria optimisation problem

$$(f, g, \mathcal{M}_{\mathrm{IR}})$$

All conditions of Theorem 2.3 item 4 are satisfied. Thus, the Pareto frontier is given by a piecewise linear, downwards sloping functions whose graph is in $\mathbb{R}^2_+$. This gives the main characterisation result in Mennle and Seuken 2016.

Remark 2.5 (On the meaning of f)**.** Mennle and Seuken 2016 consider the following concept: A mechanism is ε-strategyproof if and only if

$$\sum_{i=1}^{m} u_i(m)(\varphi(P_i', P_{-i}) - \varphi(P_1, \dots, P_n)) \leq \varepsilon,$$
$$\forall i \in N, \forall (P_1, \dots, P_n) \in (\mathcal{P}(M))^n, P_i' \in \mathcal{P}(M) \forall (u_1, \dots, u_n) \in U_{P_1} \times \dots \times U_{P_n}. \quad (4)$$

To use this as an objective for multicriteria optimisation, one might use the function

$$f'(\varphi) = \max_{\substack{(P_1,\dots,P_n)\in(\mathcal{P}(M))^n \\ P_i' \in \mathcal{P}(M) \\ (u_1,\dots,u_n)\in U_{P_1}\times\dots\times U_{P_n}}} \sum_{i \in N} \sum_{i=1}^{m} u_i(m)(\varphi(P_i', P_{-i}) - \varphi(P_1, \dots, P_n))$$

whose maximum is unfortunately not finite due to the lowest line. One can show however for f from (3)

Proposition 2.6. $f = f'$

Hence one can apply all the results from this section.

The concept of ε-strategyproofness has been used in Birrell and Pass 2011. They argue that the concept is sensible as in many situations deviations that only yield a very small gain are not undertaken, as lying is assumed to be associated with psychological costs. Mennle and Seuken 2016 do not point out that this concept does not allow for a revelation principle: Relaxing the assumptions on lying in the sense of ε-strategyproofness means that there is no concept of equilibrium underlying a mechanism. This is an unpleasant feature of the model in Mennle and Seuken 2016.

Both f and g are anonymous and neutral. Then Proposition 2.4 tells us that each point $\mathbf{x}$ on the Pareto frontier can be realised (in the sense of $y \in f^{-1}(\{\mathbf{x}\})$) by a mechanism that is anonymous and neutral. This is what Mennle and Seuken 2016 call *anonymity* resp *neutrality for free*.

Mennle and Seuken 2016 finish with an open problem: They conjecture, that the Pareto frontier of $(\mathcal{M}_{\mathrm{IR}}, f, g)$ converges for $m, n \to \infty$ to $(\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)$. As they do not state what they *mean* by convergence, this might be clarified. Tatur 2005 presents a limit result on Pareto frontiers. We unify both by considering *convergence in Hausdorff metric* as a concept of convergence for Pareto frontiers.

3. Limits of Pareto Frontiers

So far, we thought of multi-criteria optimisation problem as fixed. This in particular means that the number of agents in a mechanism design problem must be constant. Often, however, one is interested in the limit of large markets. The natural setup then is to consider a sequence $(f_1^i, \ldots, f_k^i, \Omega^i)_{i\in\mathbb{N}}$ of k-criteria optimisation problems. Write again $f^i = (f_1^i, \ldots, f_k^i)$. This gives sequences

$$P_i = P_{(f_1^i,\ldots,f_k^i,\Omega^i)} \qquad\qquad F_i = F_{(f_1^i,\ldots,f_k^i,\Omega^i)}$$

In the following section we propose that convergence of Pareto frontiers can be formulated as convergence in Hausdorff metric of the feasible sets F_i. It is not clear whether such a convergence for the Pareto frontier itself can be established.

3.1. Hausdorff Metric Convergence

Let us define the distance on sets we are going to consider. We assume now that $y_j\mathbf{e}_j \in f^i(\Omega) \subseteq \mathbb{R}_+^k$, $\forall j = 1, \ldots, k$ and $i \in \mathbb{N}$.

Definition 3.1 (Hausdorff metric)**.** Let A, B be sets in a metric space (M, d) (such as $\mathbb{R}^k$). Then define

$$d(A, B) = \inf\{\varepsilon \geq 0 |\ B_\varepsilon(B) \supseteq A \wedge B_\varepsilon(A) \supseteq B\}$$

One has the following:

Proposition 3.2 (Blaschke Selection Theorem)**.** *The set of compact subsets of a compact metric space endowed with Hausdorff metric is a* compact *metric space.*

For a proof see e.g. Price 1940. In general, the feasible sets are not compact (as they contain arbitrarily high values). If we can bound the values of one desideratum at which all other desiderata can be satisfied perfectly (i.e. we can bound y_i from above uniformly in i), then sequences of feasible sets will only differ on a compact set. Then one can restrict to this compact set and apply Proposition 3.2. This we will use in the proof of Proposition 3.4 below.

If elements in sequences only differ on a compact set, some kind of monotonicity of the sequence (e.g. inclusion of Pareto frontiers in one another) will suffice to show existence of limit Pareto frontiers which might be interesting in the future.

Last, we show that results such as Theorem 1 from Tatur 2005 can also be phrased in the language of Pareto frontiers.

3.2. Convergence in Tatur 2005

Tatur 2005 considers a model of stochastic sale of multiple goods with Bayesian truth-telling: In contrast to Mennle and Seuken 2016 it is with transferable utility. Call $m \in \mathbb{N}$ the *market size*. There are a number $|S| = k_1 \cdot m$ of sellers and $|B| = k_2 m$ of buyers that have quasilinear utility functions and types v_i are sampled from distribution functions F resp. G[5] for buyers resp. sellers. One considers again random mechanisms (in the set $\mathcal{M}_m$). This defines the set of mechanisms one might consider. Define

$$\operatorname{deficit}(q,t) \coloneqq \max\left\{\mathbb{E}\left[\sum_{i\in B\cup S} t_i(v)\right], 0\right\}$$

for a profile of valuations v, the *deficit* and

$$\mathbb{E}\left[\sum_{i\in B\cup S} v_i q_i(v) - \sum_{i\in S} v_i\right],$$

the gains from trade. One would like to minimise the distance to the efficient mechanism, that is, the one that allocates the good to the $k_1 m$ agents with the highest valuations. In accordance with Tatur 2005 we call this mechanism q^{eff}. Then the inefficiency of a mechanism is given by

$$\operatorname{inefficiency}(q,t) \coloneqq \left\{\mathbb{E}\left[\sum_{i\in B\cup S} v_i q_i^{\mathrm{eff}}(v) - \sum_{i\in S} v_i\right] - \mathbb{E}\left[\sum_{i\in B\cup S} v_i q_i(v) - \sum_{i\in S} v_i\right], 0\right\}$$

Also define

$$\operatorname{inefficiency}_m(q,t) = m \operatorname{inefficiency}(q,t)$$

All IC and IR constraints are linear. As the type space is continuous, we may only use Theorem 2.3 item 1. This means, that we know that the Pareto Frontier is the graph of a convex function that is finite in 0 and has a zero. Denote by $\mathcal{M}_m$ the set of feasible mechanisms in this setup. Tatur 2005 shows the following:

Theorem 3.3 (Tatur 2005)**.** *There are explicit constants $A, B \in \mathbb{R}^+$ such that for any $x \in \mathbb{R}$ and $(M^m)_{m\in\mathbb{N}}$, $M_m \in \mathcal{M}_m$ and* $\operatorname{deficit}(M^m) \leq -x, \forall m \in \mathbb{N}$.

$$\liminf_{m\to\infty} \operatorname{inefficiency}_m(M^m) \geq (A(x+B)^2)^+,$$

where $(\cdot)^+$ denotes the positive part. This bound is tight in the sense that for any $x \in \mathbb{R}_+$ there is a sequence of mechanisms (M^m) such that $\operatorname{deficit}(M^n) \to -x$ *and* $\lim_{m\to\infty} \operatorname{inefficiency}_m(M^m) = (A(x+B)^2)^+$.

We show that this can also be phrased as convergence of Pareto Frontiers. This statement easily implies Theorem 3.3.

[5] that are assumed to have bounded support and such that $k_1 F(p) = k_2(1 - G(p))$ has a unique root p^* w.r.t. p. F and G are assumed to be twice continuously differentiable around p^*.

Proposition 3.4. *Let F_m be the feasible set of the 2-criteria optimisation problem $(\mathcal{M}_m, \text{deficit}, \text{inefficiency}_m)$ with any fixed $m \in \mathbb{N}$. Then*

$$F_m \to \{(x, (A(-x+B)^2)^+), x \in [0, \infty)\} + \mathbb{R}_+^2$$

w.r.t Hausdorff metric.

Proposition 3.4 easily implies Theorem 3.3. The converse requires a little argument.

4. Conclusion

We gave some technical tools for further analysis of Pareto frontiers. We gave easier proofs of the major results in Mennle and Seuken 2016 and gave an alternative formulation of one of the main theorems in Tatur 2005.

Open problems are to prove the conjecture of Mennle and Seuken 2016 mentioned at the end of subsection 2.2 and to establish the existence of Pareto Frontiers (so far, one can only guarantee subsequential convergence) in more general settings.

A. Proofs

Proof of Theorem 2.3. We first prove item 1. Let $(f_1, \ldots, f_k, \Omega)$ be a k-criteria optimisation problem. Then $f(\Omega) + \mathbb{R}_+^k$ is convex: Indeed, let $\lambda \in (0, 1)$ and $x, y \in f(\Omega)$. Let $\tilde{x}$ and $\tilde{y}$ be their pre-images under f. Then for any $i = 1, \ldots, k$,

$$\lambda f_i(\tilde{x}) + (1-\lambda) f_i(\tilde{y}) \geq f_i(\lambda \tilde{x} + (1-\lambda)\tilde{y}) \in f(\Omega)$$

by the convexity of f_i. This implies $\lambda f(\tilde{x}) + (1-\lambda) f(\tilde{y}) = \lambda x + (1-\lambda) y \in f(\Omega) + \mathbb{R}_+^k$. This is the graph of a function $\mathbb{R}^{k-1} \to \mathbb{R}$: Indeed, fix $x \in \mathbb{R}^{k-1}$. If the Pareto frontier was not the graph of a function, then there would necessarily need to be two points (x, y_1) und (x, y_2) in this set. W.l.o.g. $y_1 < y_2$. But then (x, y_2) Pareto dominates (x, y_1), which is a contradiction to the definition of the Pareto frontier. This function is convex, as the feasible set is convex.

For item 2, we will need a further notion, the *projection* of a polyhedron.

Definition A.1. Let $A \subset \mathbb{R}^k$ a set. Let $I \subseteq \{1, \ldots, \ell\}$. Denote as an abuse of notation for $a \in \mathbb{R}^I$ and $b \in \mathbb{R}^{\{1,\ldots,k\} \setminus I}$ by (a, b) the vector that has values of x_i at indices that belong to I and y_i for other indices. Then define

$$\pi_I(A) = \{a \in \mathbb{R}^I \mid \exists b \in \mathbb{R}^{\{1,\ldots,k\} \setminus I} : (a, b) \in A\}$$

We have the following:

Lemma A.2. *Let $A \subseteq \mathbb{R}^k$ be a polyhedron and $I \subseteq \{1, \dots, k\}$. Then $\pi_I(A)$ is a polyhedron.*

We do not prove this result here. An excellent explanation can be found in Schrijver 1998, section 12.2, pp. 155–157, which works by the so-called *Fourier-Motzkin elimination.*

Now we can start proving item 2: Let $\Omega = \{\mathbf{Ax} \leq \mathbf{b}\}$. Let $f_i = \max_{k \in A_i} \ f_i^k$ for a finite set A_i. Then:

$$\begin{aligned}
\mathbf{x} \in f(\Omega) + \mathbb{R}_+^k &\iff \exists \mathbf{y} \in \Omega \colon \mathbf{x} \geq f(\mathbf{y}) \\
&\iff \exists \mathbf{y} \in \Omega \forall i = 1, \dots, k, \ell \in A_i \colon x_i \geq f_i^\ell(\mathbf{y}) \\
&\iff \mathbf{x} \in \pi_{\{1,\dots,k\}} \left(\{(\mathbf{x}, \mathbf{y}) | \ \mathbf{Ay} \ \leq \mathbf{b} \text{ and } f_i^k(\mathbf{y}) \leq x_i \forall i, k\} \right)
\end{aligned}$$

But as we assume all f_i^k to be linear, this shows that $f(\Omega) + \mathbb{R}_+^k$ is the projection of $\{(\mathbf{x}, \mathbf{y}) | \ \mathbf{Ay} \ \leq \mathbf{b}$ and $f_i^k(\mathbf{y}) \leq x_i \forall i, k\}$, a polyhedron, hence itself a polyhedron.

The item 3 is a special case of item 2.

For item 4 we know by item 3 that the feasible set is of the form

$$F_{(f_1,\dots,f_k,\Omega)} = \operatorname{conv}(\{\mathbf{x}_1, \dots, \mathbf{x}_m\}) + \mathbb{R}_+^2 \ \subseteq \mathbb{R}_+^2 \tag{5}$$

It is well known that the pareto minimal points P of a set $\Omega \subseteq \mathbb{R}^k$ are given by

$$P = \{\mathbf{x} \in \Omega | \exists \mathbf{0} \neq \lambda \geq 0 : x \in \arg\min_{x \in \Omega} \lambda^T \mathbf{x}\} \tag{6}$$

Assume w.l.o.g. that $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is minimal with (5). Then the vertices[6] of $F_{(f_1,\dots,f_k,\Omega)}$ are given by $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$.

Assume furthermore that $\mathbf{x}_i$ are increasing in their first coordinate. Then they must be decreasing in their second coordinate. Indeed, otherwise $\mathbf{x}_i \leq \mathbf{x}_j \iff \mathbf{x}_j \in \mathbf{x}_i + \mathbb{R}_+^2$, contradicting the assumed minimality. Then all edges of $F_{(f_1,\dots,f_k,\Omega)}$ are line segments. All vertices have to be the endpoint of an edge (otherwise $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ would not have been minimal). There can only be line segments between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$. Indeed, one easily sees that in any other case, three line segments $\operatorname{conv}\{\mathbf{x}_i, \mathbf{x}_j\}$, $\operatorname{conv}\{\mathbf{x}_j, \mathbf{x}_k\}$ and $\operatorname{conv}\{\mathbf{x}_k, \mathbf{x}_l\}$ for $i < k < j < l$ would exist, which can easily been shown to contradict the convexity of the feasible set.

□

[6] Let $F = \{\mathbf{x} | \ \mathbf{Ax} \geq \mathbf{b}\}$. Then the faces of F are sets of the form $\{\mathbf{x} | \ \mathbf{A}_1\mathbf{x} \geq \mathbf{b}_1, \mathbf{A}_2\mathbf{x} \geq \mathbf{b}_2\}$, where $\mathbf{A}_1 \cup \mathbf{A}_2$ are a partition of the rows of $\mathbf{A}$ and $\mathbf{b}_1 \cup \mathbf{b}_2$ is the corresponding partition of the rows of $\mathbf{b}$. One calls one-dimensional faces *vertices* and two-dimensional faces *edges*. An easy observation is that all linear functionals (hence in particular the non-negative ones) are optimised at faces.

Proof of Proposition 2.4. Let $\mathbf{x} \in P_{(f_1,\dots,f_k,\Omega)}$, that is a Pareto minimal point and let $x \in f^{-1}(\mathbf{x})$. Then by assumption

$$g.x \in \mathcal{M} \qquad\qquad f_i(g.x) = f_i(x), \forall i = 1, \dots, k.$$

Consider the element $m = \frac{1}{|G|}\sum_{g\in G} g.x$ ($\in \mathcal{M}$, as $\mathcal{M}$ is convex), which has the property that

$$h.m = \frac{1}{|G|}\sum_{g\in G}(hg).x = \frac{1}{|G|}\sum_{k\in G} k.x = m,$$

as the function $G \ni g \mapsto hg \ \in G$ is bijective for any $h \in G$, its inverse given by $g \mapsto h^{-1}g$. Furthermore

$$f_i(m) = f_i\left(\sum_{g\in G}\frac{1}{|G|}(g.x)\right) \leq \sum_{g\in G}\frac{1}{|G|}f_i(g.x) = \sum_{g\in G}\frac{1}{|G|}f_i(x) = f_i(x), \forall i = 1, \dots, k$$

where $\leq$ follows from the convexity of f_i and the last equality from the invariance of f_i. But then $f(m) \leq f(x)$ coordinate-wise and $m \in \mathcal{M}$ Thus m is also in the Pareto frontier. □

Proof of Proposition 2.6. Clearly, $f' \leq f$, as the maximum is over a smaller set in the further.

For the converse, fix any $i \in N$, $(P_1, \dots, P_n)$ and P_i' and denote the objective (writing $u = (u_1, \dots, u_n)$)

$$m(u) \coloneqq \sum_{i=1}^{m} u_i(m)(\varphi(P_i', P_{-i}) - \varphi(P_1, \dots, P_n))$$

which is a linear function (we suppress the dependance on $i \in N$, $(P_1, \dots, P_n)$ and P_i'). One easily sees that $U_{P_i} = \mathrm{conv}(\hat{U}_{P_i})$. Hence also $U_{P_1} \times \cdots \times U_{P_n} = \mathrm{conv}(\hat{U}_{P_1} \times \cdots \hat{U}_{P_n})$. Hence, we can write for any $u \in U_{P_1} \times \cdots \times U_{P_n}$

$$u = \sum_{u'\in\hat{U}_{P_1}\times\cdots\hat{U}_{P_n}} \lambda_{u'}u'.$$

But then

$$m(u) = m\left(\sum_{u'\in\hat{U}_{P_1}\times\cdots\hat{U}_{P_n}} \lambda_{u'}u'\right) = \sum_{u'\in\hat{U}_{P_1}\times\cdots\hat{U}_{P_n}} \lambda_{u'}m(u') \leq \max_{u'\in\hat{U}_{P_1}\times\cdots\hat{U}_{P_n}} u'$$

Hence also

$$\begin{aligned} f(\varphi) = &\max_{\substack{(P_1,\dots,P_n)\in(\mathcal{P}(M))^n \\ P_i'\in\mathcal{P}(M) \\ i\in N}} \max_{u\in U_{P_1}\times\cdots U_{P_n}} m(u) \\ &\leq \max_{\substack{(P_1,\dots,P_n)\in(\mathcal{P}(M))^n \\ P_i'\in\mathcal{P}(M) \\ i\in N}} \max_{u\in\hat{U}_{P_1}\times\cdots\hat{U}_{P_n}} m(u) = f'(\varphi), \end{aligned} \tag{7}$$

which finishes the proof. □

Proof of Proposition 3.4. We prove this proposition in three steps: First we show that we can restrict convergence to a compact set. Second, we show that all converging subsequences of (F_n) converge to $\{(x, (A(-x+B)^2)^+), x \in [0, \infty)\} + \mathbb{R}_+^2$. Third, we conclude that this implies convergence.

First, consider the two functions Tatur 2005 minimises in his bicriteria optimisation. Tatur 2005, Corollaries 2 and 3 show that there is a sequence of mechanisms (the double auctions with transaction fees) that realise points $(0, z_n)$ resp. $(w_n, 0)$ on the pareto frontier for $z_n \to z$ and $w_n \to w$ for explicit constants z and w. But this means — as convergent sequences are bounded — that we can give a uniform bound C on $|z_n|$ and $|w_n\,|$. Hence, there is a compact set — namely $\{(x, y)|\ |x|, |y| \leq C\}$ such that

$$F_m \cap C^C = F_n \cap C^C, \forall m, n \in \mathbb{N}.$$

Then, convergence in Hausdorff distance of (F_n) is equivalent to convergence in Hausdorff distance of $(F_n \cap C)$. Thus, we may w.l.o.g. assume that the space we are considering is compact.

Second, consider a converging subsequence $(F_{n_k})_{k \in \mathbb{N}}$. It is well known, that if a sequence is convergent in Hausdorff distance, then one has

$$\lim_{k \to \infty} F_{n_k} = \{x \in \Omega | \exists (x_{n_k})_{k \in \mathbb{N}}, x_{n_k} \in F_{n_k}, x_{n_k} \to x\} \tag{8}$$

(see e.g. Henrikson 1999) so the limit set is the set of all limit points of sequences in the sets in the sequence. We would like to show that for $F := \{(x, (A(-x+B)^2)^+), x \in [0, \infty)\} + \mathbb{R}_+^2$

$$F = \lim_{k \to \infty} F_{n_k} \tag{9}$$

for any subsequence $(n_k)_{k \in \mathbb{N}}$. We show that the two parts of Theorem 3.3 correspond the two inclusions of (9). Consider first the inclusion $\subseteq$. This means that there is a sequence of mechanisms that converge to an arbitrary point on F. As we can shift each sequence by a positive vector, we may assume that $x \in F$ is Pareto minimal, hence in $\{(x, (A(-x+B)^2)^+), x \in [0, \infty)\}$. We know from the second part of Theorem 3.3 that such a sequence exists in the feasible set. The inclusion of $\supseteq$ is a mere reformulation of the first part of Theorem 3.3.

Third, we would like to conclude from this that there is Hausdorff convergence. Assume the sequence was not convergent to F. Then we can find an ε and a subsequence $(F_{n_{k_\ell}})$ such that $d(F_{n_{k_\ell}}, F) > \varepsilon$ for any $\ell \in \mathbb{N}$. As by the first part we may assume that the space is compact, we may assume that there is a subsequence of $F_{n_{k_\ell}}$ that is convergent. By the second part, this subsequence must converge to F. This is a contradiction and shows that $F_n \to F$ in Hausdorff metric. □

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